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Master Equation for Quantum Brownian Motion Derived by Stochastic Methods

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The master equation for a linear open quantum system in a general environment is derived using a stochastic approach. This is an alternative derivation to that of Hu, Paz, and Zhang, which was based on the direct computation of path integrals, or to that of Halliwell and Yu, based on the evolution of the Wigner function for a linear closed quantum system. We first show by using the influence functional formalism that the reduced Wigner function for the open system coincides with a distribution function resulting from averaging both over the initial conditions and the stochastic source of a formal Langevin equation. The master equation for the reduced Wigner function can then be deduced as a Fokker-Planck equation obtained from the formal Langevin equation.

1. INTRODUCTION

Open quantum systems are of interest in condensed matter physics (Caldeira and Leggett, 1983a; Leggett *et al*., 1987), quantum optics (Walls and Milburn, 1994), quantum measurement theory (Zurek, 1981, 1982), nonequilibrium field theory (Calzetta and Hu, 1988, 2000; Calzetta *et al.*, 2000a; Stephens *et al.*, 1999), quantum cosmology (Habib, 1990; Habib and Laflamme, 1990; Paz and Sinha, 1991, 1992), and semiclassical gravity (Calzetta and Verdaguer, 1999; Hu, 1989). An open quantum systm consists of a subset of degrees of freedom, whose dynamics one is interested in, within a larger closed quantum system undergoing unitary evolution (Davies, 1976). This subsystem of interest is simply called the "system" whereas the remaining degrees of freedom constitute the "environment." In general, the evolution of the system will be nonunitary and even non-Markovian.

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A typical example of an open quantum system is the quantum Brownian motion (QBM) model, which consists of a single massive particle interacting with an infinite set of independent harmonic oscillators with a Gaussian initial state (Ford *et al*., 1963; Rubin, 1960, 1961; Zwanzig, 1959). The coupling may be linear both in the system and environment variables or may be nonlinear in some or all of these variables. The frequencies of the environment oscillators are distributed according to a prescribed spectral density function, the simplest case corresponding to the so-called ohmic environment. Part of the interest of the linear systems is that they are in many cases exactly solvable and detailed studies of different aspects of open quantum systems can be performed. One of the issues that have received much attention in recent years is environment-induced decoherence as a mechanism to understand the transition from the quantum to the classical regime (Zurek, 1991, 1993).

Certain useful information for an open quantum system is contained in the *master equation*. The master equation is a differential equation for the time evolution of the reduced density matrix of an open quantum system. The master equation for linear coupling and ohmic environment at high temperature was first deduced by Caldeira and Leggett (1983b), it was extended to arbitrary temperature by Unruh and Zurek (1989), and it was finally obtained for a general environment (i.e., for an arbitrary spectral density function) by Hu, Paz, and Zhang using path integrals (Hu *et al*., 1992). This result was then extended to the case of nonlinear coupling by treating the interaction perturbatively up to quadratic order (Hu *et al*., 1993).

The *reduced Wigner function* is defined from the reduced density matrix by an integral transform (Hillary *et al*., 1984; Wigner, 1932). This function is similar in many aspects to a distribution function in phase space, although it is not necessarily positive definite, and the dynamical equation it satisfies is similar to the *Fokker-Planck equation* for classical statistical systems (Gardiner, 1983; Risken, 1989; Wax, 1954). This equation is, of course, entirely equivalent to the master equation for the reduced density matrix and we will often also refer to it as the master equation. Halliwell and Yu exploited the fact that the Wigner function for a linear closed quantum system evolves according to the classical equations of motion to obtain the equation satisfied by the reduced Wigner function (Halliwell and Yu, 1996). The reduced density matrix has been used to study decoherence induced by the environment (Caldeira and Leggett, 1985; Giulini *et al*., 1996; Hu *et al*., 1992, 1993; Joos and Zeh, 1985; Paz and Zurek, 1993; Unruh and Zurek, 1989; Zurek *et al*., 1993). The Wigner function has also been used in studies of emergence of classicality induced by an environment (Paz *et al*., 1993), especially in quantum cosmology (Habib, 1990; Habib and Laflamme, 1990; Paz and Sinha, 1991, 1992).

Langevin type of equations (Sancho and San Miguel, 1979; Zwanzig, 1973) as a suitable tool to study the semiclassical limit have been used recently in

semiclassical gravity and cosmology (Calzetta *et al*., 1997; Calzetta and Hu, 1994; Calzetta and Verdaguer, 1999; Campos and Verdaguer, 1996; Hu and Matacz, 1995; Hu and Sinha, 1995; Martín and Verdaguer, 1999a–c, 2000). In inflationary cosmology they have been used to describe the stochastic effect on the inflaton field (Calzetta and Gonorazky, 1997; Calzetta and Hu, 1995; Goncharov and Linde, 1986; Goncharov *et al*., 1987; Habib, 1992; Habib and Kandrup, 1992; Kiefer *et al*., 1998a,b; Kiefer and Polarski, 1998; Linde, 1986; Matacz, 1997a,b; Mijic, 1990; Polarski and Starobinsky, 1996; Rey, 1987; Starobinsky, 1986) or the stochastic behavior of large-scale gravitational perturbations (Roura and Verdaguer, 1999a, 2000), which is important for cosmological structure formation. So far, in the functional approach the Langevin equation has been mainly restricted to describe the classical or semiclassical limit. See, however, Ford *et al*. (1988) for a quantum version of the Langevin equation in operator language.

A closer look at the influence functional, nevertheless, reveals that a formal Langevin equation can be extracted from this functional independently of the existence of a classical limit at least for quadratic influence actions. This Langevin equation is used to show that the reduced Wigner function can be written as a formal phase-space distribution function associated to a stochastic process (Calzetta *et al*., 2000b) [as earlier suggested in Anglin and Habib (1996)]. The master equation governing its time evolution is then deduced as the corresponding Fokker-Planck equation.

The plan of the paper is the following: In section 2 we briefly summarize the essential concepts and results of the influence functional formalism for linear open quantum systems. In section 3 we show how the reduced Wigner function for the system can be expressed as an average over the different realizations of a stochastic process. This result is used in section 4 to give an alternative derivation of the master equation for a general environment. Finally, we summarize and discuss our results in section 5.

2. INFLUENCE FUNCTIONAL FORMALISM AND MASTER EQUATION FOR LINEAR OPEN QUANTUM SYSTEMS

Let us first review a QBM model as an example of linear open quantum system. We consider a harmonic oscillator of mass M, the "system," coupled to a bath of independent harmonic oscillators of mass m, the "environment." For simplicity, let us assume that the system and environment are linearly coupled. The action for the whole set of degrees of freedom is defined by:

$$S[x, \{q_j\}] = S[x] + S[\{q_j\}] + S_{\text{int}}[x, \{q_j\}], \tag{2.1}$$

where the terms on the right-hand side correspond to the action of the system, the environment and the interaction term respectively. They are given by:

$$S[x] = \int dt \left(\frac{1}{2} M \dot{x}^2 - \frac{1}{2} M \Omega^2 x^2 \right), \tag{2.2}$$

$$S[\{q_j\}] = \sum_j \int dt \left(\frac{1}{2} m \dot{q}_j^2 - \frac{1}{2} m \omega_j^2 q_j^2 \right), \tag{2.3}$$

$$S_{\rm int}[x, \{q_j\}] = \sum_j c_j \int dt x(t) q_j(t) = \int_0^\infty d\omega \frac{2m\omega}{\pi c(\omega)} I(\omega) \int dt x(t) q(t;\omega), \tag{2.4}$$

where we introduced the spectral density $I(\omega) = \sum_j \pi c_j^2 (2m\omega_j)^{-1} \delta(\omega - \omega_j)$ in the last equality, $c(\omega)$ and $q(t;\omega)$ are functions such that $c(\omega_j) = c_j$ and $q(t;\omega_j) = q_j(t)$, c_j being system-environment coupling parameters, and Ω and ω_j are, respectively, the system and environment oscillator frequencies. When no special form is assumed for the spectral density $I(\omega)$, this is usually referred to as a general environment. One of the most common particular cases is the so-called Ohmic environment, defined by $I(\omega) \sim \omega$ (some high frequency cut-off may be sometimes naturally introduced).

The reduced density matrix for an open quantum system is defined from the density matrix ρ of the whole system by tracing out the environment degrees of freedom

$$\begin{aligned} \rho_r(x_f, x_f', t_f) &= \int \prod_j dq_j \rho(x_f, \{q_j\}, x_f', \{q_j\}, t_f) \\ &= \int dx_i \, dx_i' J(x_f, x_f', t_f; x_i, x_i', t_i) \rho_r(x_i, x_i', t_i), \end{aligned} \tag{2.5}$$

where the last equation gives the evolution of the reduced density matrix by means of the propagator J, which is defined in a path integral representation by

$$J(x_f, x_f', t_f; x_i, x_i', t_i) = \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}x \int_{x'(t_i)=x_i'}^{x'(t_f)=x_f'} \mathcal{D}x' \, e^{i(S[x]-S[x']+S_{IF}[x,x'])/\hbar}, \tag{2.6}$$

where $S_{IF}[x, x']$ is the influence action introduced by Feynman and Vernon (1963). When the system and the environment are initially uncorrelated, i.e., when the initial density matrix factorizes ($\hat{\rho}(t_i) = \hat{\rho}_r(t_i) \otimes \hat{\rho}_e(t_i)$, where $\hat{\rho}_r(t_i)$ and $\hat{\rho}_e(t_i)$ mean,

respectively, the density matrix operators of the system and the environment at the initial time) the influence functional, defined by $F[x, x'] = \exp(iS_{IF}[x, x'])/\hbar$, can be expressed in the following way:

$$F[x, x'] = \prod_j \int dq_j^{(f)}\, dq_j^{(i)}\, dq_j'^{(i)} \int_{q_j(t_i)=q_j^{(i)}}^{q_j(t_f)=q_j^{(f)}} \mathcal{D}q_j \times \int_{q_j'(t_i)=q_j'^{(i)}}^{q_j'(t_f)=q_j^{(f)}} \mathcal{D}q_j' \exp\left[\frac{i}{\hbar}(S[\{q_j\}] - S[\{q_j'\}] + S[x, \{q_j\}] - S[x', \{q_j'\}])\right] \cdot \rho_e\left(\left\{q_j^{(i)}\right\}, \left\{q_j'^{(i)}\right\}, t_i\right). \tag{2.7}$$

When the initial density matrix for the environment $\rho_e(\{q_j^{(i)}\}, \{q_j'^{(i)}\}, t_i)$ is Gaussian, the path integrals can be exactly performed and one obtains (Caldeira and Leggett, 1983b; Feynman and Hibbs, 1965; Feynman and Vernon, 1963):

$$S_{IF}[x, x'] = -2\int_{t_i}^{t_f} ds \int_{t_i}^{s} ds' \Delta(s) D(s, s') X(s') + \frac{i}{2}\int_{t_i}^{t_f} ds \times \int_{t_i}^{t_f} ds' \Delta(s) N(s, s') \Delta(s'), \tag{2.8}$$

where $X(s) \equiv (x(s) + x'(s))/2$ and $\Delta(s) \equiv x'(s) - x(s)$. The kernels $D(s, s')$ and $N(s, s')$ are called the dissipation and noise kernel, respectively.

For environments consisting of an infinite number of oscillators it is especially convenient to rewrite the first term on the right-hand side of Eq. (2.8) as

$$\int_{t_i}^{t_f} ds \int_{t_i}^{t_f} ds' \Delta(s) H_{\text{bare}}(s, s') X(s'), \tag{2.9}$$

where we defined $H_{\text{bare}}(s, s')$ as formally equivalent to $-2D(s, s')\theta(s - s')$. Being the product of two distributions the latter expression is not well defined in general and suitable regularization and renormalization may be required; see (Roura and Verdaguer, 1999b) for details. The local divergences present in $H_{\text{bare}}(s, s') = H(s, s') + H_{\text{div}}\delta(s - s')$ can be canceled by suitable counterterms Ω_{div} in the bare frequency of the system $\Omega = \Omega_{\text{ren}} + \Omega_{\text{div}}$. From now on we will consider that this infinite renormalization, if necessary, has already been performed so that both Ω_{ren} and $H(s, s')$ are free of divergences.

From Eqs. (2.5) and (2.6) a differential equation for the system's reduced density matrix known as the master equation can be derived. The expression for a

general environment was first obtained by Hu, Paz, and Zhang using a path integral approach (Hu *et al*., 1992) [see (Paz, 1994) for a slightly different derivation]:

$$i\hbar\frac{\partial\rho_r}{\partial t} = -\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2}\right)\rho_r + \frac{1}{2}M\Omega^2(x^2 - x'^2)\rho_r$$
$$+ \frac{1}{2}M\delta\Omega^2(t)(x^2 - x'^2)\rho_r - i\hbar A(t)(x - x')\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)\rho_r$$
$$+ \hbar B(t)(x - x')\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'}\right)\rho_r - iMC(t)(x - x')^2\rho_r, \quad (2.10)$$

where the functions $\delta\Omega^2(t)$, $A(t)$, $B(t)$, and $C(t)$ represent a frequency shift, a dissipation factor, and two diffusive factors, respectively. For explicit expressions of these functions see section 4. An alternative representation for the system reduced density matrix is the reduced Wigner function $W_r(X, p, t)$ defined as

$$W_r(X, p, t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} d\Delta\, e^{ip\Delta/\hbar}\,\rho_r(X - \Delta/2, X + \Delta/2, t). \quad (2.11)$$

It follows immediately that the master equation (2.10) can be written in the following equivalent form:

$$\frac{\partial W_r}{\partial t} = \{H_R, W_r\}_{PB} + 2A(t)\frac{\partial(pW_r)}{\partial p} + \hbar B(t)\frac{\partial^2 W_r}{\partial q\partial p} + \hbar MC(t)\frac{\partial^2 W_r}{\partial p^2}, \quad (2.12)$$

where $\{H_R, W_r\}_{PB} \equiv -(p/M)\partial W_r/\partial q + M\Omega_R^2(t)q\partial W_r/\partial p$ with $\Omega_R^2(t) = \Omega^2 + \delta\Omega^2(t)$. This equation was directly derived by Halliwell and Yu (1996) exploiting the fact that the Wigner function for the whole closed quantum system evolves according to the classical equations of motion. Note that Eq. (2.12) is formally similar to the Fokker-Planck equation for a distribution function.

3. STOCHASTIC FORMAL EXPRESSION FOR THE REDUCED WIGNER FUNCTION

In this section we show that the reduced Wigner function can be written as a formal distribution function for some stochastic process [see Calzetta *et al*. (2000b) for a detailed exposition]. This will be the key starting point in the derivation of the master equation given in the next section.

In order to find an explicit expression for the reduced density matrix (2.5) at a time t_f, we need to compute the path integrals appearing in Eq. (2.6) for the reduced density matrix propagator. From now on we will consider $\hbar = 1$. After integrating the system action by parts and performing the Gaussian path integral

for $\Delta(t)$ with Δ_i and Δ_f fixed, we obtain

$$\int_{X_i}^{X_f} \mathcal{D}X \int_{\Delta_i}^{\Delta_f} \mathcal{D}\Delta\, e^{i\Delta\cdot L\cdot X}\, e^{-\frac{1}{2}\Delta\cdot N\cdot\Delta} = \left(\det\frac{N}{2\pi}\right)^{-\frac{1}{2}} \int_{X_i}^{X_f} \mathcal{D}X\, e^{-\frac{1}{2}(L\cdot X)\cdot N^{-1}\cdot(L\cdot X)}, \tag{3.1}$$

where $L(t,t') \equiv M(\frac{d^2}{dt'^2} + \Omega_{\text{ren}}^2)\,\delta(t-t') + H(t,t')$. Taking into account the surface terms arising from the integration by parts of the system action and definition (2.11) for the reduced Wigner function, the result of the integration over Δ_i gives

$$\begin{aligned}\rho_r(X_f - \Delta_f/2,\, X_f + \Delta_f/2,\, t_f) &= 2\pi\left(\det\frac{N}{2\pi}\right)^{-\frac{1}{2}} \int_{-\infty}^{\infty} dX_i \int_{X_i}^{X_f} \mathcal{D}X\, e^{-\frac{1}{2}(L\cdot X)\cdot N^{-1}\cdot(L\cdot X)} \\ &\quad\times e^{-iM\dot{X}_f\Delta_f} W_r(X_i,\, M\dot{X}_i,\, t_i).\end{aligned} \tag{3.2}$$

The next step to perform is the following functional change:

$$X(t) \to \{X_i = X(t_i),\, p_i \equiv M\dot{X}_i = M\dot{X}(t_i),\, \xi(t) = (L\cdot X)(t)\}. \tag{3.3}$$

Note that with this change the function $X(t)$ gets substituted by the initial conditions $(X_i,\, p_i)$ and the function $\xi(t)$ in the path integration. It is important to note that at this point the function $\xi(t)$ is not a stochastic process but just a function over which a path integral is performed. The functional change (3.3) is invertible as can be explicitly seen:

$$\{X_i,\, p_i,\, \xi(t)\} \to X(t) = X_o(t) + \int_{t_i}^{t} dt' G_{\text{ret}}(t,t')\xi(t'), \tag{3.4}$$

where $G_{\text{ret}}(t',t'')$ is the retarded (i.e., $G_{\text{ret}}(t',t'') = 0$ for $t' \le t''$) Green function for the linear integro-differential operator associated to the kernel $L(t,t')$, and $X_{\text{inh}}(t) = \int_{t_i}^{t} dt' G_{\text{ret}}(t,t')\xi(t')$ is a solution of the inhomogeneous equation $(L\cdot X_{\text{inh}})(i) = \xi(t)$ with initial conditions $X_{\text{inh}}(t_i) = 0$ and $\partial X_{\text{inh}}(t')/\partial t'|_{t'=t_i} = 0$. On the other hand, $X_o(t)$ is a solution of the homogeneous equation $(L\cdot X_o)(t) = 0$, with initial conditions $X_o(t_i) = X_i$ and $\dot{X}(t_i) = p_i/M$. Since the change is linear, the Jacobian functional determinant will be a constant (this can be clearly seen by skeletonizing the path integral). After performing the functional change, we obtain

$$\begin{aligned}\rho_r(X_f - \Delta_f/2,\, X_f + \Delta_f/2,\, t_f) = K \int_{-\infty}^{\infty} dX_i \int_{-\infty}^{\infty} dp_i \int \mathcal{D}\xi\, \delta(X(t_f) - X_f) \\ \times e^{-\frac{1}{2}\xi\cdot N^{-1}\cdot\xi} e^{-iM\dot{X}(t_f)\Delta_f} W_r(X_i,\, p_i,\, t_i),\end{aligned} \tag{3.5}$$

where the delta function $\delta(X(t_f) - X_f)$ was introduced to restrict the functional integral $\int \mathcal{D}\xi$ with free ends, in order to take into account the restriction on the final points of the allowed paths for the integral $\int^{X_f} \mathcal{D}X$ appearing in Eq. (3.2). The

contribution from the Jacobian has been included in the constant K. By demanding the reduced density matrix to be normalized, i.e., that $Tr\rho_r(t_f) = 1$, provided that the initial Wigner function is properly normalized, this constant can be determined to be

$$K = \left[\int \mathcal{D}\xi \; e^{-\frac{1}{2}\xi\cdot N^{-1}\cdot\xi}\right]^{-1} = [\det(2\pi N)]^{-\frac{1}{2}}. \tag{3.6}$$

Finally, using the definition (2.11) for the Wigner function and the fact that $(2\pi)^{-1}\int_{-\infty}^{\infty} d\Delta_f \, e^{ip_f\Delta_f} \, e^{-iM\dot{X}(t_f)\Delta_f} = \delta(M\dot{X}(t_f) - p_f)$, we get an expression for the reduced Wigner function

$$W_r(X_f, p_f, t_f) = K\int_{-\infty}^{\infty} dX_i \int_{-\infty}^{\infty} dp_i \int \mathcal{D}\xi\,\delta(X(t_f) - X_f)\delta(M\dot{X}(t_f) - p_f)$$
$$\times\, e^{-\frac{1}{2}\xi\cdot N^{-1}\cdot\xi} W_r(X_i, p_i, t_i), \tag{3.7}$$

which can be written in the following suggestive way:

$$W_r(X_f, p_f, t_f) = \langle\langle\delta(X(t_f) - X_f)\delta\,(M\dot{X}(t_f) - p_f)\rangle_\xi\rangle_{X_i,p_i}, \tag{3.8}$$

where

$$\langle\ldots\rangle_\xi \equiv [\det(2\pi N)]^{-\frac{1}{2}}\int \mathcal{D}\xi\ldots e^{-\frac{1}{2}\xi\cdot N^{-1}\cdot\xi}, \tag{3.9}$$

$$\langle\ldots\rangle_{X_i,p_i} \equiv \int_{-\infty}^{\infty} dX_i \int_{-\infty}^{\infty} dp_i \ldots W_r(X_i, p_i, t_i). \tag{3.10}$$

Thus the reduced Wigner function can be interpreted as an average over a Gaussian stochastic process $\xi(t)$ with $\langle\xi(t)\rangle_\xi = 0$ and $\langle\xi(t)\xi(t')\rangle_\xi = N(t, t')$ as well as an average over the initial conditions characterized by a distribution function $W_r(X_i, p_i, t_i)$. It is only after formally interpreting $\xi(t)$ as a stochastic process characterized by Eq. (3.9) that the equation defining $\xi(t)$ in the functional change (3.3)

$$(L\cdot X)(t) = \xi(t), \tag{3.11}$$

can be regarded as a Langevin equation. We insist that, in general, Eq. (3.11) is not meant to describe the actual trajectories of the system, but it should rather be regarded as a formal tool. We should also remark that $X(t_f)$ and $\dot{X}(t_f)$ in Eq. (3.8) correspond to solutions of the Langevin equation (3.11) for a given realization of $\xi(t)$, and that X_f and p_f are coordinates of a point in phase space.

Note, in addition, that although $W_r(X_i, p_i, t_i)$ is real, which follows from the hermiticity of the density matrix, and properly normalized, in general it is not positive everywhere and, thus, cannot be considered as a probability distribution. The fact that the Wigner function cannot be interpreted as a phase-space probability density is crucial since most of the nonclassical features of the quantum state are tightly related to the Wigner function having negative values. For instance,

a coherent superposition state is typically characterized by the Wigner function presenting strong oscillations with negative values in the minima (Giulini *et al.*, 1996; Paz *et al.*, 1993), which are closely connected to interference terms.

4. FROM LANGEVIN TO FOKKER-PLANCK: DERIVATION OF THE MASTER EQUATION

As mentioned above there is a simple one-to-one correspondence between any density matrix and the associated Wigner function introduced in (2.11). Taking this correspondence into account, the equation satisfied by the reduced Wigner function is equivalent to the master equation satisfied by the reduced density matrix.

Equation (3.8) shows that the reduced Wigner function can be interpreted as a formal distribution in phase space. By deriving it with respect to time and using the Langevin-type equation in (3.11), one can obtain a Fokker-Planck differential equation describing the time evolution of the system's reduced Wigner function.

The derivation of the Fokker-Planck equation from the Langevin equation with local dissipation is well understood [see Sancho and San Miguel (1979)]. However, in our case the existence of nonlocal dissipation makes it convenient to review the main steps. Let us begin by computing $\partial W_r/\partial t$ from expression (3.8),

$$\begin{aligned}\frac{\partial W_r(X,\,p,\,t)}{\partial t} &= \langle\langle \dot{X}(t)\delta'(X(t)-X)\delta(M\dot{X}(t)-p)\rangle_\xi\rangle_{X_i,\,p_i} \\ &\quad + \langle\langle \delta(X(t)-X)M\ddot{X}(t)\delta'(M\dot{X}(t)-p)\rangle_\xi\rangle_{X_i,\,p_i} \\ &= -\frac{p}{M}\frac{\partial W_r(X,\,p,\,t)}{\partial X} - \frac{\partial}{\partial p}\langle\langle \delta(X(t)-X)M\ddot{X}(t) \\ &\quad \times \delta(M\dot{X}(t)-p)\rangle_\xi\rangle_{X_i,\,p_i}, \end{aligned} \tag{4.1}$$

where the fact that $\dot{X}(t)$, $\partial/\partial X(t)$, and $\partial/\partial\dot{X}(t)$ may be replaced by p/M, $-\partial/\partial X$, and $-\partial/\partial p$, respectively, since they are multiplying the delta functions, was used in the second equality. Let us now concentrate on the expectation value appearing in the last term and recall the expectation values defined in (3.9)–(3.10). We will consider the Langevin-type equation

$$(L\cdot X)(t') = \xi(t'), \tag{4.2}$$

corresponding to the functional change (3.3) and substitute the corresponding expression for $M\ddot{X}(t)$ so that the last expectation value in (4.1) can be written as

$$\begin{aligned} -M\Omega_{\text{ren}}^2 X W_r(X,\,p,\,t) + \left\langle\left\langle\left(-\int_{t_i}^{t} dt\, H(t,\,t')X(t') + \xi(t)\right)\right.\right. \\ \left.\left.\times\,\delta(X(t)-X)\delta(M\dot{X}(t)-p)\right\rangle_\xi\right\rangle_{X_i,\,p_i}. \end{aligned} \tag{4.3}$$

Any solution of Eq. (4.2) can be written as

$$X(t') = X_h(t') + \int_{t'}^{t} dt'' \tilde{G}_{\rm adv}(t', t'')\xi(t''), \tag{4.4}$$

where $X_h(t')$ is a solution of the homogeneous equation $(L \cdot X)(t') = 0$ such that $X_h(t) = X$, $\dot{X}_h(t) = p/M$ and $\tilde{G}_{\rm adv}(t', t'')$ is the advanced (i.e., $\tilde{G}_{\rm adv}(t', t'') = 0$ for $t' \geq t''$) Green function for the linear integro-differential operator associated to the kernel $L(t, t')$. The particular solution of the inhomogeneous Eq. (4.2)

$$\tilde{X}_{\rm inh}(t') = \int_{t'}^{t} dt'' \tilde{G}_{\rm adv}(t', t'')\xi(t'') \tag{4.5}$$

has boundary conditions $\tilde{X}_{\rm inh}(t) = 0$, $\partial \tilde{X}_{\rm inh}(t')/\partial t'|_{t'=t} = 0$. Both $X_h(t')$ and $\tilde{G}_{\rm adv}(t', t'')$ can be expressed in terms of the homogeneous solutions $u_1(t')$ and $u_2(t')$, which satisfy $u_1(t_i) = 1$, $u_1(t) = 0$ and $u_2(t_i) = 0$, $u_2(t) = 1$, respectively:

$$X_h(t') = X\left(u_2(t') - \frac{\dot{u}_2(t)}{\dot{u}_1(t)}u_1(t')\right) + \frac{(p/M)}{\dot{u}_1(t)}u_1(t'), \tag{4.6}$$

$$\tilde{G}_{\rm adv}(t', t'') = -\frac{1}{M}\frac{u_1(t')u_2(t'') - u_2(t')u_1(t'')}{\dot{u}_1(t'')u_2(t'') - \dot{u}_2(t'')u_1(t'')}\theta(t'' - t'). \tag{4.7}$$

We use the advanced propagator so that there is no dependence on the initial conditions at time $t' = t_i$ coming from the homogeneous solution but just on the final conditions at time $t' = t$, i.e., on those the Fokker-Planck equation is written in terms of. Using expression (4.4) the first term within the expectation value appearing in Eq. (4.3) can be reexpressed as

$$\begin{aligned}\int_{t_i}^{t} dt H(t, t')\langle\langle X(t')\delta(X(t) - X)\delta(M\dot{X}(t) - p)\rangle_\xi\rangle_{X_i, p_i} \\ = \int_{t_i}^{t} dt' H(t, t')X_h(t')W_r(X, p, t) + \int_{t_i}^{t} dt' \int_{t'}^{t} dt'' H(t, t') \\ \times \tilde{G}_{\rm adv}(t', t'')\langle\langle \xi(t'')\delta(X(t) - X)\delta(M\dot{X}(t) - p)\rangle_\xi\rangle_{X_i, p_i}.\end{aligned} \tag{4.8}$$

The first term on the right-hand side can in turn be written as

$$-(M\delta\Omega(t)X + 2A(t)p)\, W_r(X, p, t), \tag{4.9}$$

where

$$\delta\Omega(t) = \frac{1}{M}\int_{t_i}^{t} dt' H(t, t')[u_2(t') - (\dot{u}_2(t)/\dot{u}_1(t))u_1(t')], \tag{4.10}$$

$$A(t) = \frac{1}{2}(M\dot{u}_1(t))^{-1}\int_{t_i}^{t} dt' H(t, t')u_1(t'). \tag{4.11}$$

In order to find an expression for $\langle \xi(t')\delta(X(t)-X)\delta(M\dot{X}(t)-p)\rangle_\xi$ we use Novikov's formula for Gaussian stochastic processes (Novikov, 1965), which corresponds essentially to use (3.9) and functionally integrate by parts with respect to $\xi(t)$,

$$\langle \xi(t')F(t;\xi]\rangle_\xi = \int_{t_i}^{t} dt'' N(t', t'')\, \langle \delta F(t;\xi]/\delta\xi(t'')\rangle_\xi . \tag{4.12}$$

We then obtain the following expression:

$$\begin{aligned}
&\langle \xi(t')\delta(X(t)-X)\delta(M\dot{X}(t)-p)\rangle_\xi \\
&\quad = \int_{t_i}^{t} dt''' \int_{t_i}^{t} dt'' N(t', t'') \left\langle \left(\frac{\delta X(t''')}{\delta\xi(t'')} \frac{\delta}{\delta X(t''')} + \frac{\delta \dot{X}(t''')}{\delta\xi(t'')} \frac{\delta}{\delta \dot{X}(t''')} \right) \right. \\
&\qquad \left. \times \delta(X(t)-X)\delta(M\dot{X}(t)-p) \right\rangle_\xi \\
&\quad = \int_{t_i}^{t} dt''' \int_{t_i}^{t} dt'' N(t', t'')\delta(t'''-t) \left\langle -\left(\frac{\delta X(t''')}{\delta\xi(t'')} \frac{\partial}{\partial X} + M \frac{\delta \dot{X}(t''')}{\delta\xi(t'')} \frac{\partial}{\partial p} \right) \right. \\
&\qquad \left. \times \delta(X(t)-X)\delta(M\dot{X}(t)-p) \right\rangle_\xi \\
&\quad = \int_{t_i}^{t} dt'' N(t', t'') \left\langle -\left(\frac{\delta X(t)}{\delta\xi(t'')} \frac{\partial}{\partial X} + M \frac{\delta \dot{X}(t)}{\delta\xi(t'')} \frac{\partial}{\partial p} \right) \right. \\
&\qquad \left. \times \delta(X(t)-X)\delta(M\dot{X}(t)-p)_\xi \right\rangle,
\end{aligned} \tag{4.13}$$

where we used again the presence of the delta functions to substitute the functional derivatives $\delta/\delta X(t''')$ and $\delta/\delta\dot{X}(t''')$ by $-\delta(t'''-t)\cdot\partial/\partial X$ and $-\delta(t'''-t)\cdot M\cdot\partial/\partial p$, respectively, in the second equality. Functionally differentiating with respect to $\xi(t'')$ expression (3.4) for $X(t)$ and analogously for $\dot{X}(t)$ we get

$$\frac{\delta X(t)}{\delta\xi(t'')} = G_{\text{ret}}(t, t''), \tag{4.14a}$$

$$\frac{\delta \dot{X}(t)}{\delta\xi(t'')} = \frac{\partial}{\partial t} G_{\text{ret}}(t, t''), \tag{4.14b}$$

which after substitution into (4.14) leads to

$$\begin{aligned}
&\langle \xi(t')\delta(X(t)-X)\delta(M\dot{X}(t)-p)\rangle_\xi \\
&\quad = -\int_{t_i}^{t} dt'' N(t', t'') \left(G_{\text{ret}}(t, t'') \frac{\partial}{\partial X} + M \frac{\partial G_{\text{ret}}(t, t'')}{\partial t'} \frac{\partial}{\partial p} \right) W_r(X, p, t).
\end{aligned} \tag{4.15}$$

The retarded Green function can also be expressed in terms of the solutions of the homogeneous equation $u_1(t)$ and $u_2(t)$, which were previously introduced, as

$$G_{\rm ret}(t', t'') = \frac{1}{M}\frac{u_1(t')u_2(t'') - u_2(t')u_1(t'')}{\dot{u}_1(t'')u_2(t'') - \dot{u}_2(t'')u_1(t'')}\theta(t' - t''). \tag{4.16}$$

Note that it is important to use now the expression in terms of the retarded propagator $G_{\rm ret}$ and the initial conditions X_i and p_i (at time $t' = t_i$), since the "final" conditions $X(t)$ and $M\dot{X}(t)$ depend on $\xi(t'')$ (for $t'' < t$). Putting all the terms together, i.e., (4.3), (4.9), and (4.16), we reach the final expression for (4.1):

$$\frac{\partial W_r}{\partial t} = \{H_R, W_r\}_{PB} + 2A(t)\frac{\partial(pW_r)}{\partial p} + B(t)\frac{\partial^2 W_r}{\partial X \partial p} + MC(t)\frac{\partial^2 W_r}{\partial p^2}, \tag{4.17}$$

where $\delta\Omega(t)$ and $A(t)$ are given by Eqs. (4.11) and (4.12), and

$$B(t) = \int_{t_i}^{t} dt''' N(t, t''')G_{\rm ret}(t, t''') - \int_{t_i}^{t} dt' H(t, t') \int_{t'}^{t} dt'' \tilde{G}_{\rm adv}(t', t'') \times \int_{t_i}^{t} dt''' N(t'', t''')G_{\rm ret}(t, t'''), \tag{4.18}$$

$$C(t) = \int_{t_i}^{t} dt''' N(t, t''')\frac{\partial G_{\rm ret}(t, t''')}{\partial t} - \int_{t_i}^{t} dt' H(t, t') \int_{t'}^{t} dt'' \tilde{G}_{\rm adv}(t', t'') \times \int_{t_i}^{t} dt''' N(t'', t''')\frac{\partial G_{\rm ret}(t, t''')}{\partial t''}. \tag{4.19}$$

The last two expressions were obtained by combining the second term within the expectation value appearing in (4.3) and the second term on the right-hand side of Eq. (4.9). It should be taken into account that if we put back the $\hbar$'s, there appears one with every noise kernel in Eqs. (4.19) and (4.20).

The expressions (4.11), (4.12), (4.19), and (4.20) for $\delta\Omega(t)$, $A(t)$, $B(t)$, and $C(t)$, respectively, coincide exactly with those of Halliwell and Yu (1996), which are in turn equivalent to those obtained in Hu *et al*. (1992). Thus, this derivation of the master equation based on a stochastic description for the system is an alternative to those given previously (Halliwell and Yu, 1996; Hu *et al*., 1992; Paz, 1994) and is, of course, in agreement with their results.

5. DISCUSSION

In this paper we have considered the stochastic description of a linear open quantum system to give an alternative derivation of the corresponding master equation. We have shown that the reduced Wigner function can be written as a formal distribution function for a stochastic process characterized by a Langevin-type equation. The master equation has then been deduced as the corresponding

Fokker-Planck equation for the stochastic process. This derivation can be extended to the case of nonlinear interaction between system and environment by computing perturbatively the influence action up to quadratic order and even to the case of a general potential for the system (Calzetta *et al*., 2001).

It should be pointed out that whereas one can derive the Fokker-Planck equation from the Langevin equation, the opposite is not possible in general. One can always consider Langevin equations with stochastic sources characterized by different noise kernels which, nevertheless, lead to the same Fokker-Planck equation and, thus, the same master equation. This can be argued from the expressions obtained in the derivation of the Fokker-Planck equation. Let us consider, for simplicity, the situation corresponding to local dissipation. A local contribution to the noise gives no contribution to $B(t)$, but it does contribute to $C(t)$ as can be seen from Eqs. (4.19) and (4.20) taking into account that $G_{\rm ret}(t, t) = 0$ and $\partial G_{\rm ret}(t', t)/\partial t' \,|_{t'=t} = M^{-1}$. Thus, one can always choose any noise kernel that gives the desired $B(t)$ and then add the appropriate local contribution to the noise kernel to get the desired $C(t)$ keeping $B(t)$ fixed. Note that changing the noise kernel does not change $A(t)$. To illustrate the fact that there exist different noise kernels giving the same $B(t)$, as was stated above, one may consider the particular case corresponding to the weak dissipation limit so that $G_{\rm ret}(t, t') \sim (M\Omega)^{-1} \sin \Omega(t - t')\theta(t - t')$. To see that a different $\tilde{N}(t, t')$ giving the same $B(t)$ as $N(t, t')$ exists reduces then to show that there is at least one nontrivial function $\nu(s, t) = \tilde{N}(t, t') - N(t, t')$ (with $s = t - t'$) such that for any t one has $\int_0^t ds \sin(\Omega s)\nu(s, t) = 0$, which can be shown to be the case.

The fact that different Langevin equations lead to the same master equation reflects that the former contains more information than the latter. To be more precise, what we showed was that a Langevin equation contains in general more information that the corresponding Fokker-Planck equation. To extend this assertion to the master equation, one should make sure that different Langevin equations leading to the same Fokker-Planck equation can be obtained from an influence functional. Indeed this can be shown to be the case provided that one considers general Gaussian initial states for the environment. The inequivalence between the Langevin equation and the master equation can be qualitatively understood in the following way. In the influence functional it is only the evolution of the environment degrees of freedom that is traced out. Of course, having integrated over all the possible quantum histories for the environment, no correlations in the environment can be obtained. Nevertheless, since the system is interacting with the environment, non-Markovian correlations for the system at different times may in general persist. On the other hand, when considering either the reduced density matrix or its propagator, also the system evolution, except for the final state, is integrated out. Consequently, information on non-Markovian time correlations for the system is no longer available. Thus, only when the system's reduced dynamics is Markovian, i.e., the influence functional is local in time, we expect that

the Langevin equation and the master equation contain the same information. In particular, for a Gaussian stochastic source, as in our case, the Langevin equation contains the information about the system correlations at different times, which the Fokker-Planck equation cannot in general account for. Only in the case in which the dynamics generated by the Langevin equation is Markovian one can compute the correlation functions just from the solutions of the Fokker-Planck equation or, equivalently, the master equation for the propagator $J(x_2, x_2', t_2; x_1, x_1', t_1)$; see Eq. (2.6). The key point is the fact that the propagator for the reduced density matrix only factorizes when the influence action is local. See Calzetta *et al*. (n.d.) for a detailed argument on this point.

It is important to note that for a closed quantum system the evolution determined by the time evolution operators $U(t_2, t_1)$ obtained from the Schrödinger equation is always unitary and, thus, also Markovian. That is why the Schrödinger equation suffices to get the correlation functions for a closed quantum system. On the contrary, for an open quantum system the evolution is nonunitary and, provided the influence action is nonlocal, not even Markovian.

Finally, we should insist on the fact that, although we have exploited the formal description of open quantum systems in terms of stochastic processes, a classical statistical interpretation is not always possible. Thus, although the Wigner function is a real and properly normalized function providing a distribution for the initial conditions of our formal stochastic processes, it is not a true probability distribution function in the sense that it is not positive definite in general. In fact, this property is crucial for the existence of quantum coherence for the system. Nevertheless, even though the Langevin equation does not in general describe actual classical trajectories of the system, it is still a very useful tool to compute quantum correlation functions (Calzetta *et al*., 2000b) or even as an intermediate step to derive the master equation.

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